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STABILITY OF INTERPHASE BOUNDARIES IN SOLID ELASTIC MEDIA*

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An appraoch is developed for the stability analysis of the equilibrium of heterogeneous thermodynamic systems with phase transformation surfaces of the first kind, based on investigations of the non-negative-definiteness of the second energy variation. A derivation is given of explicit second-variation formulas of the corresponding functionals for cases of systems with solid single-component phases under coherent transformations (of martensitic type) and transitions with slip. The general procedure is illustrated by the example of a stability investigation for transitions with slip in the approximation of an asymptotic of low "intrinsic" strain in an isothermal system with isotropic elastic phases.

The numerous forces recently presupposed resulted in substantial progress in the understanding of the fundamental concepts of the mechanics of a continuous medium in application to the problem of phase transformations in solids. The explicit perception of the inconsistency of the concept of a scalar chemical potential of a solid raised the question of finding correct phase equilibrium conditions for transformations in a solid substance, which, in turn resulted in a more detailed and clear classification of phase transitions of the first kind in a solid and in a development of representations about different chemical potential tensors. Meanwhile sequential specific results referring to equilibrium *stability* do not exist for different transitions of the first kind in heterogeneous systems. It is natural to start the study of the equilibrium stability of heterogeneous systems with phase transition surfaces with the static underlying the Gibbs approach /1/ associated with computations of the second variations of the appropriate thermodynamic functionals in the neighbourhood of the equilibrium state. The variation concept should be consistent internally with the physical nature of the interfacial boundaries under consideration.

Below we describe the results of an investigation of the necessary conditions for stability (the sufficient conditions for instability) for cases of coherent phase transformations and phase transitions with slip; some of these results were given in /2, 3/. The stability conditions obtained are conditions of non-negative-definiteness of the properly understood second variations of the corresponding energy functions (in connection with the second variation criterion in continuum problems, see /4/). A brief derivation of the second variation of the energy functional is given initially in the neighbourhood of the equilibrium state in a set of allowable configurations dictated by the physical nature of the transformation being studied. Furthermore, the question of non-negative-definiteness of the second variation is reduced to confirmation of the non-negativity of spectral values of a linear homogeneous system of partial differential equations with appropriate boundary conditions. Then the spectral problem for the case of the asymptotic form of a small "natural" deformation of the transformation is reduced, in this approximation, to an explicit stability analysis for the simplest phase symmetry results in completely clear results. This is illustrated by an explicit stability analysis of the interphase boundary in an isothermal system with isotropic phases during transitions with slip: here an equation is obtained for the critical deformations (the neutral equilibrium condition) which are of the order of the natural deformation of the transformation.

1. Equilibrium and stability conditions of heterogeneous systems with coherent transformation surfaces.

A complete thermodynamic analysis of the equilibrium and stability of heat-insulated systems on the basis of the Gibbs principles /1/ in the case of simple elastic phases with no external force fields present is reduced to an investigation of the minimum of the total internal energy E for a fixed total entropy S

$$E=\int_{\omega}' d\omega m e(u_{k|l}, \eta),\quad S=\int_{\omega}' d\omega m\eta \tag{1.1}$$

When investigating coherent transformations, the Lagrangian description of a continuous medium is used everywhere in this paper: $u_k(x)$ are the components of the displacement vector at a point with the Lagrangian coordinates x^i on the basis of the initial configuration; the

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Latin subscript after the vertical bar denotes covariant differentiation on the basis of the metric initial configuration tensor x_{ij}, x^{ij}, used also for "juggling" the spatial Lagrangian indices: m is the mass density in the initial homogeneous configuration (common to both phases); $e(u_{k|l}, \eta)$ is the dependence of the specific (per unit mass) internal energy of simple elastic phases on the displacement gradients $u_{k|l}$ and the specific entropy η; ω is the domain occupied by the system in the initial configuration, and $\int'$ is the symbol of the sum of the integrals over the smooth parts of the system.

In the case of coherent transformations the displacement field on the interphase boundary γ is continuous by definition: $[u^i] = 0$ ($[a] = a_+ - a_-$); by virtue of this, the following compatibility relationships for discontinuities of the derivatives

$$\begin{aligned}
&[u_{i|j}] = h_i n_j, \quad \left[\frac{\partial u_i}{\partial \tau}\right] = -ch_i, \quad h_i = [u_{i|k}]\, n^k \qquad (1.2)\\
&[u_{i|jk}] = H_i n_j n_k + 2h_{i|\alpha} n_{(j} x_{k)}{}^{\alpha} - h_i b_{jk}, \quad H_i = [u_{i|kl}]\, n^k n^l\\
&\left[\frac{\partial u_{i|j}}{\partial \tau}\right] = -H_i c n_j + \frac{\delta h_i}{\delta \tau} n_j - (ch_i)_{|\alpha}\, x_j{}^{\alpha}\\
&\left[\frac{\partial^2 u_i}{\partial \tau^2}\right] = H_i c^2 - 2c\frac{\delta h_i}{\delta \tau} - h_i \frac{\delta c}{\delta \tau}, \quad x^i_{\cdot\alpha} = \frac{\partial x^i(\xi, \tau)}{\partial \xi^\alpha}
\end{aligned}$$

should be satisfied for a one-parameter family of allowable displacement fields $u^i(x, \tau)$ and positions $x^i(\xi, \tau)$ of the discontinuous surface.

Here ξ^α are coordinates on the interphase boundary γ; the Greek subscript after the vertical bar denotes covariant differentiation on the basis of a metric surface-prototype tensor of the actual boundary in the initial configuration ("juggling" by the surface indices is realized by using this same tensor when examining coherent transformations); $b_{ij} = b_{\alpha\beta}(\xi, \tau)\, x_i^{\cdot\alpha} x_j^{\cdot\beta}$ ($b_{\alpha\beta}$ is the tensor of coefficients of the second quadratic form of the surface-prototype); c is the velocity of this surface in the direction of the unit normal n_i induced by the change in the variation parameter τ; $\delta/\delta\tau$ is the symbol of covariant differentiation with respect to a parameter on a moving surface, understood exactly as in /5/ (the appropriate definition for certain kinds of tensors differs substantially from those proposed earlier /6, 7/). The compatibility relationships for the discontinuities of the derivatives in the Weingarten, Appel, Levi-Civita, and Hadamard researches were improved considerably by Thomas to whom formulas (1.2) belong (see /6/, say).

Varying the functional $I = E + \Lambda S$ (Λ is the undetermined Lagrange multiplier), following /8/, we obtain

$$\frac{dI}{d\tau} = \int_\omega' d\omega m \left\{(e_\eta + \Lambda)\frac{\partial \eta(x, \tau)}{\partial \tau} - e^{ij}_{|j} \frac{\partial u_i(x, \tau)}{\partial \tau}\right\} + \int_\gamma d\gamma m \left\{c\,[e + \Lambda\eta] + \left[e^{ij} \frac{\partial u_i(x, \tau)}{\partial \tau}\right] n_j\right\} \qquad (1.3)$$

(here and henceforth, integrals over the outer boundary of the system are omitted).

Using the compatibility relationships for the discontinuities of the first order derivatives on a coherent boundary (1.2) and separating out the independent variations, we arrive at the following equilibrium conditions on the basis of (1.3):

$$\begin{aligned}
&e_\eta = \theta = -\Lambda, \quad p^{ji}_{|j} = 0 \ \text{ within the phase} \qquad (1.4)\\
&[p^{ji}]\, n_j = 0, \quad [\mu^{ij}]\, n_i n_j = 0 \ \text{ on the interphase boundary}\\
&\mu^{ij} = (e - \theta\eta)\, x^{ij} - m^{-1} p^{jk} (\delta^i_k + u_{k|}{}^i)
\end{aligned}$$

where $p^{ji} = me^{ij}$ is the Piola-Kirchhoff stress tensor, and μ^{ij} is the non-symmetric chemical potential tensor /9/. Here and henceforth, the following notation is used for the derivatives of the thermodynamic functions with respect to their arguments:

$$e_\eta = \frac{\partial e}{\partial \eta}, \quad e^{ij} = \frac{\partial e}{\partial u_{i|j}}, \quad e^{ijkl} = \frac{\partial^2 e}{\partial u_{i|j} \partial u_{k|l}} \ldots$$

Differentiating (1.3) with respect to τ and using the equilibrium equation (1.4), we arrive at the following formula for the second variation of I at $\tau = 0$ /2/:

$$\delta^2 I = \int_\omega' d\omega m \,(e^{ijkl} a_{i|j} a_{k|l} + 2e_\eta{}^{ij} a_{i|j} b + e_{\eta\eta} b^2) + \qquad (1.5)$$

$$\int_{\gamma} d\gamma\,(cx_j^{\cdot\alpha}[e^{ij}a_{i|\alpha}] + c^2x_j^{\cdot\alpha}[e^{ij}u_{i|l\alpha}]\,n^l - [e^{ij}a_i]\,x_j^{\cdot\alpha}c_{|\alpha})$$

$$a^i(x) = \frac{\partial u^i(x,0)}{\partial\tau},\quad b(x) = \frac{\partial\eta(x,0)}{\partial\tau}$$

Here a^i, b are the displacement and entropy variations, respectively. In deriving (1.5) it is necessary to use the compatibility relationships (1.2) and the properties of the $\delta/\delta\tau$-derivative, particularly the relationships

$$\frac{\delta c\,[e+\Lambda\eta]}{\delta\tau} = \frac{\delta c}{\delta\tau}[e+\Lambda\eta] + c\left[e^{ij}\frac{\partial u_{i|j}}{\partial\tau}\right] + c^2n^k[e^{ij}u_{i|jk}]$$

$$\frac{\delta}{\delta\tau}\left(\left[e^{ij}\frac{\partial u_i}{\partial\tau}\right]n_j\right) = -\left(\left[e^{ij}\frac{\partial u_i}{\partial\tau}\right]cx_j^{\cdot\alpha}\right)_{|\alpha} + \left[e^{ij}\frac{\partial u_i}{\partial\tau}\right]n_jcb_\alpha^\alpha +$$
$$c\left[e^{ij}\frac{\partial u_{i|j}}{\partial\tau}\right] + \left[e^{ij}\frac{\partial^2u_i}{\partial\tau^2} + \left(e^{ijkl}\frac{\partial u_{k|l}}{\partial\tau} + e_\eta^{ij}\frac{\partial\eta}{\partial\tau}\right)\frac{\partial u_i}{\partial\tau}\right]$$

According to the Gibbs principle, for the stability of the equilibrium of thermally and mechanically insulated systems with coherent transformation surfaces it is necessary to have the non-negativity of the functional δ^2I in displacement field variations of the particles a_i, the boundary c and the entropy b satisfying the relationships

$$[a^i] = -c\,[u^i_{|j}]\,n^j,\ \int_{\omega}' d\omega mb + \int_{\gamma} d\gamma mc\,[\eta] = 0 \tag{1.6}$$

the former of which is a consequence of the coherence condition and the latter of the fixed nature of the total entropy.

Constancy of the absolute temperture at different points of the system is not deduced in investigations of the isothermal stability, but is postulated a priori, hence the principle (of an absolute) minimum of the total free energy of the system F can be used instead of the Gibbs principle. Later, only this question will be considered (the insignificant additional difficulties associated with the presence of a constraint corresponding to the second relationship (1.6) in the case of a thermally insulated system can be taken into account as was done in the problem of thermodynamic inequalities /10/). Also assuming the equilibrium state of each of the phases to be homogeneous, and the interphase boundary to be planar, we arrive at a formula for the second variation of the total free energy of the system in the neighbourhood of an equilibrium configuration /2/

$$\delta^2F = \int_{\omega}' d\omega\psi^{ijkl}a_{i|j}a_{k|l} + \int_{\gamma} d\gamma mx_j^{\cdot\alpha}(c\,[\psi^{ij}a_{i|\alpha}] - c_{|\alpha}[\psi^{ij}a_i]) \tag{1.7}$$

where $\psi(u_{i|j}, \theta)$ is the dependence of the phase free energy density on the displacement gradients and absolute temperature Θ (which is a given parameter by virtue of the assumption).

We find the extremal values of the second free energy variation δ^2F in the set of virtual fields a^i, c satisfying the first condition in (1.6) and the normalization condition

$$G = \int_{\omega}' d\omega ma^ia_i = 1 \tag{1.8}$$

In the stable equilibrium case it is obviously necessary that these extremal values should be non-negative. The constraint (1.8) of isoperimetric type can be taken into account by the Lagrange multiplier method by going to an investigation of the absolute extremum of the functional $\Pi = \delta^2F + \pi G$ (π is an undetermined multiplier). The conditions that the first variation of the functional Π should vanish reduces to satisfying the following relationships /2/

$$\psi^{ijkl}a_{k|lj} + \pi a^i = 0 \text{ within the phase} \tag{1.9}$$

$$[\psi^{ijkl}a_{k|l}]\,n_j = c_{|\alpha}x_j^{\cdot\alpha}[\psi^{ij}],\quad x_j^{\cdot\alpha}([\psi^{ij}a_{i|\alpha}] + c_{|\alpha}[\psi^{ij}u_{i|k}]\,n^k) = [\psi^{ijkl}u_{i|p}a_{k|l}]\,n^pn_l \text{ on the interphase boundary} \tag{1.10}$$

System (1.9) and (1.10) is also supplemented by the first relationship from (1.6) and the appropriate conditions on the outer boundary. Values of the parameter π for which the mentioned linear homogeneous system has non-trivial solutions are called spectral. As was done in investigations of the thermodynamic inequalities /10/, the following assertions can be proved (which are certainly valid even for inhomogeneous equilibrium states and the case of a thermally insulated system): a) the spectral values of π are real, b) the second variation δ^2F takes a value equal to π on a non-trivial real field a^i, c belonging to the eigenvalue π and satisfying the normalization condition (1.8). Therefore, non-negativity of the spectral values of π is necessary for the isothermal stability of the system.

We say that a coherent interphase boundary is locally stable at a certain point characterized by local phase gradients $u^{\pm}_{i|j}$ if the appropriate eigenvalues π of the system formed by the first relationship of (1.6) as well as (1.9) and (1.10) corresponding to the eigenfunctions that decay exponentially in the depth of the appropriate half-spaces and are oscillatory in nature in the direction of the interphase boundary are non-negative. A study of the local stability of the interphase boundary that is similar to the study of equations with constant coefficients is substantially simpler than the stability analysis of the system as a whole and cannot ensure similar stability. At the same time, detection of the local boundary instability enables one to assess the instability of the system as a whole since the local boundary curvature and the inhomogeneity of the equilibrium configuration can be neglected for sufficiently short perturbations. Therefore, the same relation exists here as characterizes the relation between the thermodynamic stability of the material and the stability of a specific structure fabricated from it.

2. Stability of equilibrium for coherent transitions in the case of a small natural transformation deformation. For brevity, we will assume that the phases in the reference configurations (see /11, 12/) are not stressed while the system temperature corresponds to agreement between the free energy densities of the phases per unit mass. We assume the affine deformation connecting the phase reference configurations to be small

$$w_i = \varepsilon\Delta_{ij}x^j, \quad \Delta_{ij} \sim 1, \quad \varepsilon \ll 1 \tag{2.1}$$

In such a situation it is natural to expect that equilibrium configurations are found that contain both phases separated by an interphase boundary, where the physical parameters of both phases will differ slightly from the references and are represented in combination with the equation of the boundary in the form of series in a small parameter

$$v_{\pm}{}^i = \sum_{N=1}^{\infty} \varepsilon^N v^i_{N\pm}, \quad x^i(\xi, \varepsilon) = \sum_{N=0}^{\infty} \varepsilon^N x^i_N(\xi) \tag{2.2}$$

(here v^i are phase displacement fields additional to w^i).

In the situation under consideration the coefficients of the spectral problem, described by the first relationship in (1.6), as well as (1.9) and (1.10), turn out to be functions of the small parameter ε. Consequently, its solution can be sought in the form of series

$$\pi = \sum_{N=0}^{\infty} \varepsilon^N \pi_N, \quad a^i = \sum_{N=0}^{\infty} \varepsilon^N a_N{}^i, \quad c = \sum_{N=-1}^{\infty} \varepsilon^N c_N \tag{2.3}$$

Substituting (2.1)-(2.3) into the system, we reduce the spectral problem to the following form in the lowest approximation in ε:

$$\begin{aligned}
&\overline{\psi}^{*ijkl} a_{0k|lj} + \pi_0 a_0{}^i = 0 \ \text{ within the phase} \\
&[a_0{}^i] = -c_{-1}[v_{1}{}^i{}_{|j} + \Delta^i{}_j]\, n^j \ \text{ on the interphase boundary} \\
&[\overline{\psi}^{*ijkl} a_{0k|l}]\, n_{j0} = c_{-1|\alpha} x^{\alpha}_{j0} [\overline{\psi}^{*ijkl} v_{1k|l}] \\
&([\overline{\psi}^{*ijkl} v_{1k|l} a_{0i|\alpha}] + [\overline{\psi}^{*ijkl} v_{1k|l}(v_{1i|p} + \Delta_{ip})]\, n^p c_{-1|\alpha})\, x^{\alpha}_{j0} = \\
&\quad [\overline{\psi}^{*ijkl}(v_{1i|p} + \Delta_{ip})\, a_{0k|l}]\, n_0{}^p n_{j0}
\end{aligned} \tag{2.4}$$

The functions $\psi_{\pm}{}^*$ give the free energy density of the phases as a function of the quantities $v_{i|j\pm}, \theta$; the bar here denotes that the value of the appropriate derivative is calculated for $\varepsilon = 0$.

3. Necessary equilibrium and stability conditions for phase transitions with slip. Examination of the isothermal equilibrium and stability of a simple thermoelastic system in which a phase transition with slip can occur can, in the absence of external force fields, be based on an investigation of the minimum of the total free energy

$$F = \int_{\Omega}' d\Omega \rho \psi \tag{3.1}$$

In this case it is convenient to perform the description in Euler coordinates z^i: z_{ij}, z^{ij} are metric tensors of the reference system used to realize juggling by the spatial indices of the reference system, and also covariant differentiation denoted by the symbol ∇_i; U_i are Euler components of the particle displacement field, and ρ is the actual density of the substance. Covariant differentiation with respect to the coordinates ξ^α on the actual interphase boundary Σ is denoted by the symbol ∇_α (juggling of the surface indices is also realized later by using the metric tensor of the actual surface).

Bearing in mind carrying out calculations to explicit algebraic relationships, we henceforth confine ourselves to the case of isotropic non-linearly elastic simple phases. In this case the free energy density of the phases depends on the displacement gradients in a complex manner, and, depending on the convenience and purposes, can be considered as a function of the principal invariants I_M, the principal elongations Λ_M, the finite deformation tensor U_{ij}, the metric tensor of the initial configuration $z_{ij}{}^0$ etc. by virtue of the geometric relationships

$$z_{ij}{}^0 = z_{ij} - 2U_{ij}, \quad 2U_{ij} = \nabla_i U_j + \nabla_j U_i - \nabla_i U_k \nabla_j U^k \tag{3.2}$$
$$I_1 = \frac{1}{2}\frac{|z|}{|z^0|} z^{ijk} z^{pqr} z^0_{ip} z^0_{jq} z_{kr} = \Lambda_1{}^2 + \Lambda_2{}^2 + \Lambda_3{}^2, \quad |z| = |z_{ij}|$$
$$I_2 = \frac{1}{2}\frac{|z|}{|z^0|} z^{ijk} z^{pqz} z^0_{ip} z_{jq} z_{kr} = \Lambda_1{}^2\Lambda_2{}^2 + \Lambda_2{}^2\Lambda_3{}^2 + \Lambda_3{}^2\Lambda_1{}^2$$
$$I_3 = \frac{1}{6}\frac{|z|}{|z^0|} z^{ijk} z^{pqr} z_{ip} z_{jq} z_{kr} = \Lambda_1{}^2\Lambda_2{}^2\Lambda_3{}^2$$

(irrespective of the selection of the arguments to denote the free energy density, one letter will be used).

Following /9/, we use the variation technique associated with the analysis of possible particle velocities f^i and the actual boundary C that satisfy the local mass conservation condition on a singular surface

$$C\,[\rho] = [\rho f^i]\,N_i \tag{3.3}$$

in the analysis of phase transitions with slip.

Here N_i are components of the unit normal to the actual boundary.

Considering the phase free energy density as a function of the finite deformation tensor, following /9/ we reduce the first free energy variation to the form

$$\frac{dF}{d\tau} = -\int\limits_{\Omega}' d\Omega f_i \nabla_j P^{ji} + \int\limits_{\Sigma} d\Sigma\,[f_i (P^{ij} - D z^{ji})]\,N_j \tag{3.4}$$

Here P^{ji} is the Cauchy stress tensor; the scalar D characterizes the slope of the phase equilibrium curve

$$P^{ij} = \rho z^{ik} z^0_{kl} \frac{\partial\psi}{\partial U_{(lj)}}, \quad D = \frac{[\psi]}{[\rho^{-1}]}$$

From the condition that the first variation of the free energy should vanish, we arrive at the equilibrium equations /9/

$$\nabla_j P^{ji} = 0 \text{ within the phases} \tag{3.5}$$
$$P_\pm{}^{ji}|_\Sigma N_j = DN^i \text{ on the interphase boundary}$$

The isothermal stability of a configuration with a plane interface and homogeneous phase stress and strain states in the equilibrium state will be studied later. Differentiating relationship (3.4) in the neighbourhood of such a configuration and using the equilibrium conditions (3.5), we arrive at a formula for the second variation of the free energy /2/

$$\delta^2 F = \int\limits_{\Omega}' d\Omega C^{ijkl} \nabla_j f_i \nabla_l f_k + \int\limits_{\Sigma} d\Sigma z_j^{\cdot\alpha} (2C\,[d^{jk} \nabla_\alpha f_k] - [d^{ji} f^k \nabla_\alpha f_i]\,N_k) \tag{3.6}$$
$$d^{ji} = P^{ji} - Dz^{ji}, \quad C^{ijkl} = -\rho \frac{\partial\psi}{\partial U_{(lj)}} z^0_{mq} z^{kq} z^{mi} + \rho \frac{\partial^2\psi}{\partial U_{(nj)} \partial U_{(pl)}} z^0_{mn} z^0_{qp} z^{mi} z^{qk} - P^{ji} z^{kl} - P^{jk} z^{il} \tag{3.7}$$

In the derivation of the second variation formula we used the homogeneity of the equilibrium state being investigated, the properties of the $\delta/\delta\tau$-derivative and also the following relationships:

$$\frac{\delta N_i}{\delta\tau} = -z_i^{\cdot\alpha} \nabla_\alpha C, \quad \frac{\delta P^{ji}}{\delta\tau} = C^{ijkl} \nabla_l f_k$$
$$\frac{\delta}{\delta\tau}\frac{[\psi]}{[\rho^{-1}]} = \frac{1}{[\rho^{-1}]}\left[\frac{d^{ji}}{\rho}\nabla_j f_i\right]$$

4. Spectral problem to confirm the non-negativity of the second variation of the free energy. Following /2/, we consider the extremal values of the second free energy variation (3.6) in the set of kinematically allowable virtual particle velocities f^i

and the interphase boundary C satisfying the normalization condition

$$G^* = \int_\Omega d\Omega \rho f^i f_i = 1 \tag{4.1}$$

Exactly as in the case of coherent transformations, the last question is reduced to an investigation of the unconditional minimum of the functional $\Pi^* = \delta^2 F - \pi G^*$, by varying which in the set of kinematically allowable fields (i.e., those that satisfy the condition (3.3)), we obtain

$$\begin{aligned} \delta\Pi^* = & -2\int_\Omega d\Omega \delta f_i (\nabla_j C'^{ijkl}\nabla_l f_k + \pi\rho f^i) + \\ & \int_\Sigma d\Sigma \{2N_j [C'^{ijkl}\nabla_l f_k \delta f_i] + z_j^{\cdot\alpha}(2\delta C[d^{ji}\nabla_\alpha f_i] - \\ & [d^{ji}\delta f^k \nabla_\alpha f_i] N_k + [d^{ji}\delta f_i \nabla_\alpha f^k] N_k - 2[d^{ji}\delta f_i]\nabla_\alpha C)\} \\ C'^{ijkl} = & \frac{1}{2}(C^{ijkl} + C^{klij}) \end{aligned} \tag{4.2}$$

Separating the independent variations in (4,2), we arrive at the stationary conditions

$$\begin{aligned} & \nabla_j C'^{ijkl}_\pm \nabla_l f_{k\pm} + \pi\rho_\pm f_\pm{}^i = 0 \text{ within the phases} \\ & N_j C'^{ijkl}_\pm \nabla_l f_{k\pm} + z_j^{\cdot\alpha}\Big\{[d^{jk}\nabla_\alpha f_k]\frac{\rho_\pm}{[\rho]}N^i - d_\pm{}^{ji}\nabla_\alpha C - \\ & \quad \frac{1}{2}d_\pm{}^{jk}N^i\nabla_\alpha f_{k\pm} + \frac{1}{2}d_\pm{}^{ji}N_k\nabla_\alpha f_\pm{}^k\Big\} = 0 \text{ on the interphase boundary} \end{aligned} \tag{4.3}$$

Exactly as in the case of the coherent transformations, it can be shown that: a) the spectral values π of the system (4.3) (with natural conditions on the outer boundary) are real; b) in a non-trivial real field f^i belonging to the eigenvalue π and satisfying the normalization condition (4.1), the second free-energy variation (3.6) takes the value π. Therefore, for isothermal stability in phase transitions with slip, non-negativity of the spectral values of the system (4.3) is necessary. Here we do not determine the local stability of the boundary of a phase transition with slip, which is completely analogous to the case of coherent transformations.

Using (3.7), system (4.3) can be rewritten in the following equivalent form:

$$\nabla_j C^{ijkl}_\pm \nabla_l f_{k\pm} + \pi\rho_\pm f_\pm{}^i = 0 \tag{4.4}$$

$$\left(C^{ijkl}_\pm \nabla_l f_{k\pm} - \frac{1}{[\rho^{-1}]}\left[\frac{d^{kl}}{\rho}\nabla_k f_l\right] z^{ji}\right) N_j = d_\pm{}^{ji} z_j^{\cdot\alpha}\nabla_\alpha C \tag{4.5}$$

5. Local stability of the phase boundary for transitions with slip in the case of a small natural transformation deformation. We henceforth assume that the difference between the isotropic phase densities in reference configurations is quite small while the relative elongations of the phase substances during the transition from the reference configuration into the equilibrium configuration whose stability is being investigated, are close to one

$$\rho_-{}^0 = \rho_+{}^0 - \varepsilon\delta, \quad \Lambda_{M\pm} = 1 + \varepsilon e_{M\pm}, \quad \varepsilon \ll 1; \quad e_{M\pm}, \quad \delta \sim 1 \tag{5.1}$$

We will confine ourselves to the two-dimensional case by considering the components $U_\pm{}^3$, $f_\pm{}^3$ of the displacements and virtual velocities to be zero while the remaining components $U_\pm{}^1, U_\pm{}^2$ and $f_\pm{}^1, f_\pm{}^2$ are independent of z^3. We denote the horizontal coordinate z^1 in terms of x and the vertical in terms of $z = z^2$. We shall seek the solution of system (3.3), (4.4) and (4.5) in the form of the following series:

$$f_\pm{}^i = \sum_{N=0}^{\infty}\varepsilon^N f^i_{N\pm}, \quad C = \sum_{N=-1}^{\infty}\varepsilon^N C_N, \quad \pi = \sum_{N=0}^{\infty}\varepsilon^N \pi_N \tag{5.2}$$

Substituting (5.1) and (5.2) into the above-mentioned system and eliminating C by using (3.3), we arrive at relationships in the lowest approximation in ε

$$\begin{aligned} & \rho_+{}^0\pi_0 f^1_{0\pm} + (\lambda_\pm + \mu_\pm)\left(\frac{\partial f^1_{0\pm}}{\partial x} + \frac{\partial f^2_{0\pm}}{\partial z}\right) + \mu_\pm\left(\frac{\partial^2 f^1_{0\pm}}{\partial z^2} + \frac{\partial^2 f^1_{0\pm}}{\partial x^2}\right) = 0 \\ & \rho_+{}^0\pi_0 f^2_{0\pm} + (\lambda_\pm + \mu_\pm)\frac{\partial}{\partial z}\left(\frac{\partial f^1_{0\pm}}{\partial x} + \frac{\partial f^2_{0\pm}}{\partial z}\right) + \end{aligned} \tag{5.3}$$

$$\mu_\pm\left(\frac{\partial^2 f^2_{0\pm}}{\partial z^2}+\frac{\partial^2 f^2_{0\pm}}{\partial x^2}\right)=0 \text{ within the phases}$$

$$\mu_\pm\left(\frac{\partial f^1_{0\pm}}{\partial z}+\frac{\partial f^2_{0\pm}}{\partial x}\right)-\left[\frac{\partial f_0^2}{\partial x}\right]h_\pm^{11}=0 \tag{5.4}$$

$$\lambda_\pm\left(\frac{\partial f^1_{0\pm}}{\partial x}+\frac{\partial f^2_{0\pm}}{\partial z}\right)+2\mu_\pm\frac{\partial f^2_{0\pm}}{\partial z}+\left[h^{11}\frac{\partial f_0^1}{\partial x}\right]=0 \text{ on the interphase boundary}$$

The following formulas for the non-zero components of the tensors $C_\pm^{ijkl}$ in the reference configurations are used here

$$C^{1111}_{0\pm}=C^{2222}_{0\pm}=\lambda_\pm+2\mu_\pm,\quad C^{1122}_{0\pm}=C^{2211}_{0\pm}=\lambda_\pm$$
$$C^{1212}_{0\pm}=C^{2121}_{0\pm}=C^{1212}_{0\pm}=C^{1221}_{0\pm}=\mu_\pm \tag{5.5}$$

The relationships (3.2) are used here in deriving (5.5), the derivatives of the free-energy density with respect to the invariants in the reference configurations are here identified with the Lamé moduli with a computation such that equations of the classical linear theory of elasticity would be obtained on linearizing the exact non-linear equilibrium equations of an isotropic medium.

The quantities $h_\pm^{11}$ introduced into the boundary conditions (5.4) characterize the degree to which the equilibrium states of the phases are not hydrostatic

$$h_\pm^{11}=\lim_{\varepsilon\to 0}\frac{d_\pm^{11}\rho_+}{\rho_+-\rho_-}=\frac{2\mu_\pm(e_{1\pm}-e_{2\pm})}{e_{1-}+e_{2-}+\delta/\rho_+^0-e_{1+}-e_{2+}} \tag{5.6}$$

L'Hopital's rule, as well as relationships (3.2), (3.5) and (5.1) should be used in obtaining (5.6).

The solutions of (5.3) that oscillate in the x direction and decay exponentially deep in the upper (plus superscript) and lower (minus superscript) half-spaces have the form

$$f^2_{0\pm}=(B_1^\pm\exp(\mp k\xi_1^\pm z)+B_2^\pm\exp(\mp k\xi_2^\pm z))\exp(-ikx) \tag{5.7}$$
$$f^1_{0\pm}=\pm(B_1^\pm\xi_1^\pm\exp(\mp k\xi_1^\pm z)+B_2^\pm\xi_2^\pm\exp(\mp k\xi_2^\pm z))\,i\exp(-ikx)$$
$$\xi_1^\pm=(1-\pi_0/(a^2_{\parallel\pm}k^2))^{1/2},\quad \xi_2^\pm=(1-\pi_0/(a^2_{\perp\pm}k^2))^{1/2}$$

where $a_{\parallel\pm}, a_{\perp\pm}$ are the velocities of the longitudinal and transverse volume waves within the appropriate half-spaces.

Substituting soltuion (5.7) into the boundary conditions (5.4), we obtain a linear homogeneous system in $B_{1,2}^\pm$. From the condition that its discriminant vanish, we arrive at the following equation to determine the spectral value $q=\pi_0/k^2$:

$$\left\{(2-h_+)^2\xi_1^+\xi_2^+-\left(2-\frac{q}{a^2_{\perp+}}-h_+\right)^2\right\}\left\{(2+h_-)^2\xi_1^-\xi_2^--\right. \tag{5.8}$$
$$\left.\left(2-\frac{q}{a^2_{\perp-}}+h_-\right)^2\right\}+h_+^2h_-^2(1-\xi_1^+\xi_2^+)(1-\xi_1^-\xi_2^-)-$$
$$\frac{q^2}{a^2_{\perp+}a^2_{\perp-}}\left(h_+^2\frac{\mu_+}{\mu_-}\xi_2^+\xi_1^-+h_-^2\frac{\mu_-}{\mu_+}\xi_2^-\xi_1^+\right)-$$
$$2h_+h_-\left\{(2-h_+)\xi_1^+\xi_2^+-\left(2-\frac{q}{a^2_{\perp+}}-h_+\right)\right\}\times\left\{(2+h_-)\xi_1^-\xi_2^--\left(2-\frac{q}{a^2_{\perp-}}+h_-\right)\right\}=0$$

where $h_\pm=h_\pm^{11}/\mu_\pm$ are dimensionless parameters of non-hydrostaticity.

For $h_\pm=0$ Eq.(5.8) dissociates into two Rayleigh equations for surface waves in an isotropic half-space. As is well-known /13/, only positive real roots correspond to these equations. To find the neutral equilibrium equations, we should set $q=0$ in (5.8). Expanding the indeterminacy occuring here by the L'Hopital rule, we obtain

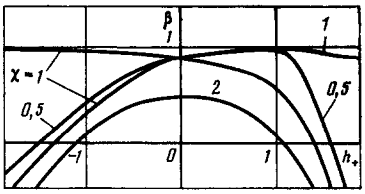
$$h_+^2\{(\varkappa_++1)(\varkappa_--1)-\chi\}+h_-^2\{(\varkappa_+-1)(\varkappa_-+1)-\chi^{-1}\}- \tag{5.9}$$
$$2h_-h_+\varkappa_+\varkappa_--4h_+\varkappa_+(\varkappa_--1)+4h_-\varkappa_-(\varkappa_+-1)+$$
$$4(\varkappa_+-1)(\varkappa_--1)=0;\quad \varkappa_\pm=a^2_{\perp\pm}/a^2_{\parallel\pm},\quad \chi=a^2_{\perp+}/a^2_{\perp-}$$

Relationships (5.8) and (5.9) are simplified significantly in the case of incompressible phases ($a_{\parallel\pm}=\infty$) when the principal forces in one of the phases (in the lower half-space, say) agree: $P_-^{11} = P_-^{22}$ (obviously $h_- = 0$ here). In the situation mentioned (5.8) and (5.9) take the form

$$\{(2-h_+)^2\sqrt{1-\beta}-(2-\beta-h_+)^2\}\{4\sqrt{1-\chi\beta}-(2-\chi\beta)^2\} - \chi^2\beta^2\sqrt{1-\beta}\,h_+^2 = 0 \tag{5.10}$$

$$h_+^2 = \frac{4}{1+\chi}, \quad \beta = \frac{q}{a_{\perp+}^2} \tag{5.11}$$

The relationship for a critical non-hydrostatic deformation (5.11) agrees with that mentioned earlier /3/.



Eq.(5.10) was solved on a computer. Represented in the figure are the dependences of the roots β on the dimensionless non-hydrostatic deformation h_+ for three values of the parameter χ (when the "instantaneous kinetics" conditions are satisfied on the interphase boundary when equilibrium succeeds in being established on it during motion in conformity with the second group of conditions (3.5), these roots can be interpreted as the ratio of the square of the surface wave velocity to the square of the transverse volume wave velocity in the "plus" phase). It is seen that the interphase boundary becomes unstable for sufficiently high non-hydrostatic deformations in this phase. The threshold value of the non-hydrostatic deformation tends to zero as the shear modulus tends to zero in the "minus" phase, which is in complete agreement with the instability detected in /3/ in a non-hydrostatically stressed solid-melt system (as is manifest for arbitrarily small non-hydrostatic stresses).

REFERENCES

1. GIBBS J.V., Papers on Thermodynamics /Russian translation/, Gostekhizdat, Moscow-Leningrad, 1950.
2. GRINFEL'D M.A., Stability of herterogeneous equilibrium in systems containing solid elastic phases, Dokl. Akad. Nauk SSSR, 265, 4, 1982.
3. GRINFEL'D M.A., Instability of the interface between a non-hydrostatically stressed elastic solid and a melt. Dokl. Akad. Nauk SSSR, 290, 6, 1986.
4. BALL J.M. and MARSDEN J.E., Quasiconvexity at the boundary, positivity of the second variation and elastic stability, Arch. Ration. Mech. and Analysis, 86, 3, 1984.
5. GRINFEL'D M.A., The ray method of calculating the wavefront intensity in non-linearly-elastic material, PMM, 42, 5, 1978.
6. THOMAS T., Plastic Flow and Fracture in Soids /Russian translation/, Mir, Moscow, 1964.
7. TRUESDELL C.A. and TOUPIN R.A., The classical field theories. Handbuch der Physik., 3/1, Springer, Berlin, 1960.
8. GRINFEL'D M.A., On thermodynamic equilibrium conditions of non-linearly elastic material phases, Dokl. Akad. Nauk SSSR, 251, 4, 1980.
9. GRINFEL'D M.A., On heterogeneous equilibrium of non-linear elastic phases and chemical potential tensors. Intern. J. Eng. Sc., 19, 7, 1981.
10. GRINFEL'D M.A., Gibbs principle and thermodynamic inequalities for non-linearly-elastic bodies. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 3, 1982.
11. GRINFEL'D M.A., On a new problem of mathematical physics associated with the problem of coherent phase transformations, Dokl. Akad. Nauk SSSR, 279, 1, 1984.
12. GRINFEL'D M.A., Asymptotic form of the small difference in densities in the coherent phase transformations problem, PMM, 49, 4, 1985.
13. LOVE A., Mathematical Theory of Elasticity /Russian translation/, Glavnaya Red. Obshchetekhn. Lit. i Nomogr., ONTI, Moscow-Leningrad, 1935.

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